On Perles' question

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Abstract

The principal aim of this paper is to determine the minimal dimension that a nerve of the cover of the sphere S^h by the open sets not containing a pair of antipodal points could have, and also to determine the minimal cardinality of such cover (or the minimal number of vertices of its nerve). In particular, our result provides the complete answer to the question posed by Micha Perles (see [6]). Our results could be seen as the extensions of the Lyusternik-Schnirel'man version of the Borsuk-Ulam theorem.

As a consequence, we also obtain the improved lower bound for the local chromatic number of certain class of graphs.

1 Introduction

The Lyusternik-Schnirel'man theorem says that if a family of open sets covers the sphere S^h, and none of them contain a pair of antipodal points, then there are at least $h + 2$ of them. It is easily verified that this statement is one of the many statements equivalent to the Borsuk-Ulam theorem. A natural question arises of determining the minimal number of these sets that contain some point of the sphere. In other words, we consider the question of determining the minimal dimension that a nerve of such a cover of the sphere could have. In [6] and [7], G. Simonyi and G. Tardos gave the complete answer to this question when h is odd, but left the ambiguity of 1 in the case when h is even. In this paper, we resolve this ambiguity.

We also discuss the minimal cardinality of the cover of the sphere satisfying the above properties, or in other words we consider the minimal number of vertices of the nerve of such a cover of the sphere of the minimal dimension. In the section 4 we give the complete answer to this question as well.

The chromatic number of a graph is a very important invariant and it attracted a lot of attention. Let us mention the nspiring result of L. Lovász solving the Kneser conjecture and determining the chromatic number of the certain class of graphs in terms of connectivity of associated simplicial complex. This result motivated a lot of research (see [4]).

The local chromatic number of a graph G is defined to be the minimum number of colors that must appear within distance 1 of a vertex of G. More formally, if we denote with $N(v)$ the neighborhood of a vertex v in a graph G (the set of vertices v is connected to), we define $\psi(G) = \min_c \max_{v \in V(G)} |\{c(u) \mid u \in N(v)\}| + 1$, where the minimum is taken over all proper colorings c of G, compare [6]. It is obvious that the local chromatic number of a graph G is smaller than its chromatic number. G. Simonyi and G. Tardos showed in their paper that if a graph G is strongly topologically t-chromatic (which is a little bit stronger assumption than to be t-chromatic), then $\psi(G) \geq \left[\frac{t}{2}\right] + 1$, (see [6]). We don't want to go too much into details here, and so refer the reader to [6] for all the definitions, statements and proofs. As a consequence of our result, we improve this lower bound for the local chromatic number of graphs by 1. There are examples described in [6] showing that this lower bound is the best possible.

2 The minimal intersections of the cover

Motivated by the related question of Matatyahu Rubin, Micha Perles formulated the following question.

Perles' question. For which h and l, the sphere S^h can be covered by open sets not containing a pair of antipodal points in such a way that no point of the sphere is contained in more than l of these sets?

For given h and l, let us denote with $Q(h, l)$ the statement that the answer to Perles' question is positive. Of course, this question is closely related to the Lyusternik-Schnirel'man version of the Borsuk-Ulam theorem. This question could be reformulated so as to determine the minimal number l such that the sphere S^h can be covered by open sets not containing a pair of antipodal points such that the intersection of any $l + 1$ of them is empty. This minimal number will be denoted by $Q(h)$.

In two of their papers G. Simonyi and G. Tardos (see [6] and [7]) treated this question. They arrived at this problem in an attempt to determine the local chromatic number of certain class of graphs. They were able to prove the following equivalence.

Theorem 2.1 ([7]) *For every h and l the statement $Q(h, l)$ is true if and only if there is a continuous map $g : S^h \to \|K\|$ to some finite simplicial complex K of dimension at most $l - 1$ satisfying $g(x) \neq g(-x)$ for all $x \in S^h$.* ∎

They proved a little bit more, namely that one could require the minimal simplices containing $g(x)$ and $g(-x)$ to be disjoint for every $x \in S^h$. This was the starting

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point in their proof that the statement $Q(h,l)$ is not true if $h \geq 2l - 1$. They also proved that $Q(h,l)$ is true if $h \leq 2l - 3$. So, only the case $h = 2l - 2$ remained open.

The statement $Q(0,1)$ (obtained when $l = 1$) is trivially true, and they proved (and independently Imre Bárány) that $Q(2,2)$ (obtained when $l = 2$) is not true.

In terms of the function $Q(h)$, they proved $\frac{h}{2} + 1 \leq Q(h) \leq \frac{h}{2} + 2$. So, for odd h we have $Q(h) = \left[\frac{h}{2}\right] + 2$ and for even h we have the ambiguity in all cases except for $h = 2$ when we know $Q(2) = 3$.

One of the aims of this note is to verify the missing case $h = 2l - 2$ completely, i.e. to determine the value $Q(h)$ for all even h. We show the following:

Theorem 2.2 *For every $l \geq 2$ the statement $Q(2l - 2, l)$ is not true, i.e. for even h we have $Q(h) = \left[\frac{h}{2}\right] + 2$.*

Using the results from [7], this theorem implies the following, improved lower bound for the local chromatic number $\psi(G)$ of the graphs which are strongly topologically t-chromatic. For the relevant definitions see [7].

Corollary 2.3 *If a graph G is strongly topologically t-chromatic, then $\psi(G) \geq \left[\frac{t}{2}\right] + 2$.* ∎

The equivalence from the theorem 2.1 reduces the proof of the theorem 2.2 to the following statement.

Lemma 2.4 *For every map $g : S^{2k} \to \|K\|$ to some finite simplicial complex K of dimension k, there is $x \in S^{2k}$ so that $g(-x) = g(x)$.*

During the preparation of this manuscript we learned that this lemma is already proved by E.V. Ščepin in [5]. His proof is based on the fact that the $(2k)$-th homotopy group $\pi_{2k}\left(\vee_{i=1}^m S_i^k\right)$ of the finite wedge of k-dimensional spheres is finite, and some clever geometrical constructions. Our proof uses completely different methods, namely the Smith theory and cohomological methods. We use the similar methods in the final section of the paper, and so we present our proof of the lemma 2.4 in the next section.

3 Proof of the main lemma

Suppose, to the contrary, that for some map g such a point $x \in S^{2k}$ does not exist. The map g induces the map $G : S^{2k} \to \|K\| \times \|K\|$ defined by $G(x) = (g(x), g(-x))$, which misses the diagonal $\Delta = \{(u,u) \mid u \in \|K\|\} \subseteq \|K\| \times \|K\|$. Let us denote with $X = (\|K\| \times \|K\|) \setminus \Delta$. So, we have the map $F : S^{2k} \to X$ defined by $F(x) = (g(x), g(-x))$. The group $\mathbb{Z}/2$ acts naturally and freely on these spaces and the map F is easily seen to be equivariant.

Throughout this paper all chain complexes and homology and cohomology groups are considered with $\mathbb{Z}/2$ coefficients which are dropped from the notation. For each

free ($\mathbb{Z}/2$)-space A, there is the Smith exact sequence of chain complexes and chain maps:

$$0 \longrightarrow C_*^s(A) \xrightarrow{\ i\ } C_*(A) \xrightarrow{\ q\ } C_*^a(A) \longrightarrow 0,$$

where the symmetric chains $C_*^s(A)$ are those chains c satisfying $\sigma c = c$ for the generator σ of the group $\mathbb{Z}/2$, and the antisymmetric chains $C_*^a(A)$ are the chains satisfying $\sigma(c) = -c$. (Consult [1].) The chain map i is the inclusion of the symmetric chains in the chain complex of all chains, and the chain map q is defined by $q(c) = c - \sigma(c)$. Moreover this exact sequence is natural, and so for the equivariant map $F : S^{2k} \to X$ of free ($\mathbb{Z}/2$)-spaces we have a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_*^s(S^{2k}) & \xrightarrow{\ i\ } & C_*(S^{2k}) & \xrightarrow{\ q\ } & C_*^a(S^{2k}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle F_s} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F_a} & & \\
0 & \longrightarrow & C_*^s(X) & \xrightarrow{\ i\ } & C_*(X) & \xrightarrow{\ q\ } & C_*^a(X) & \longrightarrow & 0
\end{array}
$$

This diagram induces the commutative diagram of the long exact sequences in cohomology. Since the spaces S^{2k} and X are $2k$-dimensional, the cohomology groups in the dimension $2k + 1$ and higher are trivial and the part of the diagram has the following form.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_a^{2k}(S^{2k}) & \xrightarrow{\ q^*\ } & H^{2k}(S^{2k}) & \xrightarrow{\ i^*\ } & H_s^{2k}(S^{2k}) & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle F_a^*} & & \uparrow{\scriptstyle F^*} & & \uparrow{\scriptstyle F_s^*} & & \\
\cdots & \longrightarrow & H_a^{2k}(X) & \xrightarrow{\ q^*\ } & H^{2k}(X) & \xrightarrow{\ i^*\ } & H_s^{2k}(X) & \longrightarrow & 0
\end{array}
$$

Since the action of the group $\mathbb{Z}/2$ on the spaces S^{2k} and X is free, the cohomology ring of symmetric chain complexes is isomorphic to the usual cohomology ring of the spaces of orbits $S^{2k}/(\mathbb{Z}/2) = \mathbb{R}P^{2k}$ and $X/(\mathbb{Z}/2)$. Moreover, the homomorphism $F_s^* : H^1(X/(\mathbb{Z}/2)) \to H^1(\mathbb{R}P^{2k})$ sends the generator $v \in H^1(X/(\mathbb{Z}/2))$ to the generator $u \in H^1(\mathbb{R}P^{2k})$. Since F_s^* is a ring homomorphism, we have $F_s^*(v^{2k}) = u^{2k} \neq 0$ in $H^{2k}(\mathbb{R}P^{2k})$.

From the exact sequence in the above diagram, it follows that the homomorphism $i^* : H^{2k}(X) \to H^{2k}(X/(\mathbb{Z}/2))$ is onto, and so there is $w \in H^{2k}(X)$ so that $i^*(w) = v^{2k}$. From the commutativity of the above diagram we have $i^*(F^*(w)) = F_s^*(i^*(w)) = u^{2k} \neq 0$.

Finally, we reach a contradiction by proving $F^*(w) = 0$. Let $j : X \to \|K\| \times \|K\|$ be the inclusion map. From the long exact sequence in cohomology of the pair $(\|K\| \times \|K\|, X)$ it follows that the map $j^* : H^{2k}(\|K\| \times \|K\|) \to H^{2k}(X)$ is onto, since $H^{2k+1}(\|K\| \times \|K\|, X) = 0$ for the dimensional reasons. So, it is enough to prove that the map $F^* \circ j^*$ is trivial.

By the Künneth formula and since the cross product could be expressed by $a \times b = p_1^*(a) \cup p_2^*(b)$ (where $p_1, p_2 : \|K\| \times \|K\| \to \|K\|$ are projections), any element in $H^{2k}(\|K\| \times \|K\|)$ could be expressed as the cup product of two elements from

$H^k(\|K\| \times \|K\|)$. (Consult [2] or [3]. Note that $\dim K = k$ and that we work with $\mathbb{Z}/2$ coefficients.) Since $F^* \circ j^*$ is a ring homomorphism, the image of any element from $H^{2k}(\|K\| \times \|K\|)$ is the cup product of two elements from $H^k(S^{2k})$. But the latter group is trivial and so is the map $F^* \circ j^*$. The proof follows. ∎

4 The minimal cardinality of the cover

In this section we address the question of the determination of the minimal total number of sets n of some cover of the sphere with the above properties, or in other words we consider the question of the determination of the minimal number of vertices of the nerve of this cover of minimal dimension. It is obvious that the sphere S^1 has a covering with three sets so that any point in the sphere is contained in at most two sets. Also, the sphere S^2 has a covering with four sets so that any point in the sphere is contained in at most three sets from the covering. For the higher dimensional spheres the question is more complicated, but the answers are more regular. We also discuss the reason for the answer in the cases of the one- and two-dimensional sphere to be different from the general scheme. We start by proving a geometrical lemma.

Lemma 4.1 *If the sphere S^h could be covered with n sets which do not contain a pair of antipodal points such that every point of the sphere is contained in at most l sets, then the sphere S^{h+1} could be covered by $n + 1$ sets which do not contain a pair of antipodal points such that every point of the sphere is contained in at most $l + 1$ sets.*

Proof: Let us consider the sphere S^h as the equatorial sphere of the sphere S^{h+1}, and its covering $\mathcal{U} = \{U_1, ..., U_n\}$ with the required properties. We describe the covering $\mathcal{V} = \{V_1, ..., V_{n-1}, V_n, V_{n+1}\}$ of the sphere S^{h+1}.

Let ε be some small positive number. For $i = 1, 2, ..., n - 1$, let V_i be the union of all arcs on the great circle containing the northern and the southern pole, and containing the points of U_i, and going from each of these points 4ε in the direction of the northern pole, and 2ε in the direction of the southern pole. Let V_n be the union of the set formed in the same way from the set U_n, and the upper hemisphere without the 3ε thick closed neighborhood of the equatorial sphere. Finally, let V_{n+1} be the lower hemisphere without the ε thick closed neighborhood of the equatorial sphere.

The reader will easily verify that the covering $\mathcal{V}$ of the sphere S^{h+1} satisfies the required properties. ∎

In the next two theorems we determine the minimal number of sets in the cover of the sphere with the required properties. The first theorem claims that a cover of given cardinality exists, and has also a geometrical proof. The second theorem states that the given cardinality is minimal and its proof is topological.

Theorem 4.2 *The odd-dimensional sphere S^{2k-1} could be covered with $2k+2$ sets which do not contain a pair of antipodal points such that every point of the sphere is contained in at most $k+1$ sets of the cover. The even-dimensional sphere S^{2k} could be covered with $2k+3$ sets which do not contain a pair of antipodal points such that every point of the sphere is contained in at most $k+2$ sets of the cover.*

Proof: The first statement of the theorem is proved in [6], and the second statement follows from the first one and from the preceding lemma. ∎

Let us denote with K_δ^2 the deleted square (here we consider the deleted product of simplicial complexes) of the complex K (i.e. the regular CW-complex whose cells are the products of two disjoint faces of K).

Theorem 4.3 *For the sphere of the dimension at least 3 the estimate obtained in the previous theorem is the best possible.*

Proof: Let us first prove that the odd dimensional sphere S^{2k-1} (when $k \geq 2$) could not be covered with $2k+1$ sets not containing a pair of antipodal points, such that every point of the sphere is contained in at most $k+1$ sets of the cover. According to the theorem 2.1, it suffices to prove that there is no mapping from the sphere S^{2k-1} to the k-dimensional skeleton K of the $(2k)$-dimensional simplex $\sigma = (v_0, v_1, ..., v_{2k})$, mapping every pair of the antipodal points of the sphere into disjoint simplices of K. (Note also the remark after the theorem 2.1.)

If such a map would exist, it would induce an $(\mathbb{Z}/2)$-equivariant map $\varphi : S^{2k-1} \to K_\delta^2$ to the deleted square of the complex K. The map φ would induce the map of the spaces of orbits $\tilde{\varphi} : \mathbb{R}P^{2k-1} \to K_\delta^2/(\mathbb{Z}/2)$. The dimension of these complexes is $2k-1$, and their top-dimensional homology groups are trivial. Namely, every top-dimensional cell (e.g. $(v_0, v_1, ..., v_k) \times (v_{k+1}, ..., v_{2k-1}, v_{2k})$) has a facet (e.g. $(v_0, v_1, ..., v_k) \times (v_{k+1}, ..., v_{2k-1})$) which is contained only in this top-dimensional cell. The same is true for the regular CW-complex of the space of orbits, and so $H_{2k-1}(K_\delta^2/(\mathbb{Z}/2)) = 0$. From this, it follows $H^{2k-1}(K_\delta^2/(\mathbb{Z}/2)) = 0$.

Now it is not difficult to obtain a contradiction in a similar way as in the preceding section. Namely, the mapping $\tilde{\varphi}^* : H^1(K_\delta^2/(\mathbb{Z}/2)) \to H^1(\mathbb{R}P^{2k-1})$ is non-trivial, i.e. it maps a generator v to the generator u. So, $\tilde{\varphi}^*(v^{2k-1}) = u^{2k-1} \neq 0$. But, $v^{2k-1} \in H^{2k-1}(K_\delta^2) = 0$, and we reached a contradiction.

Now we prove that the even dimensional sphere S^{2k} (when $k \geq 2$) could not be covered with $2k+2$ sets not containing a pair of antipodal points, such that every point of the sphere is contained in at most $k+2$ sets of the cover. Again, according to the theorem 2.1, it suffices to prove that there is no mapping from the sphere S^{2k} to the $(k+1)$-dimensional skeleton K of the $(2k+1)$-dimensional simplex $\sigma = (v_0, v_1, ..., v_{2k}, v_{2k+1})$ mapping every pair of the antipodal points of the sphere into disjoint simplices of K.

If such a map would exist, it would induce an $(\mathbb{Z}/2)$-equivariant map $\varphi : S^{2k} \rightarrow K_\delta^2$. The map φ would induce the map of the spaces of orbits $\tilde{\varphi} : \mathbb{R}P^{2k} \rightarrow K_\delta^2/(\mathbb{Z}/2)$. The dimension of these complexes is $2k$, and their top-dimensional homology groups are trivial. Namely, there are two types of the top-dimensional cells in K_δ^2 and its orbit space $K_\delta^2/(\mathbb{Z}/2)$, and those are the cells of the form $(v_0, v_1, ..., v_k) \times (v_{k+1}, ..., v_{2k}, v_{2k+1})$, and the cells of the form $(v_0, v_1, ..., v_k, v_{k+1}) \times (v_{k+2}, ..., v_{2k}, v_{2k+1})$. All the facets of the cells of the first type are also the facets of another top-dimensional cell, and this cell is of the second type. Every cell of the second type has a facet (of the form $(v_0, v_1, ..., v_k, v_{k+1}) \times (v_{k+2}, ..., v_{2k}))$ which is contained only in this top-dimensional cell. The same is true for the regular CW-complex of the space of orbits, and so $H_{2k}(K_\delta^2/(\mathbb{Z}/2)) = 0$. From this, it follows $H^{2k}(K_\delta^2/(\mathbb{Z}/2)) = 0$. Now, we reach a contradiction in the same way as for odd-dimensional spheres. ∎

Remark 4.4 If $k = 1$ in the case of the odd-dimensional sphere (i.e. if we consider the sphere S^1), the facet contained in only one top-dimensional cell does not exist, since in this case it would be of type $(v_0, v_1) \times \emptyset$ which is the empty set. Similarly, if $k = 1$ in the case of the even-dimensional sphere (i.e. if we consider the sphere S^2), the facet of the top-dimensional cell of the second type contained only in this top-dimensional cell would be of type $(v_0, v_1, v_2) \times \emptyset$ which is the empty set.

So, in the cases of the 1- and 2-dimensional spheres, the argument does not work. Moreover, as we mentioned, it is not difficult to construct the coverings in these cases with smaller number of sets. The sets in these coverings are ε neighborhoods of facets of a simplex inscribed in these spheres.

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